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# A CRACK EXTENDING NON-UNIFORMLY IN AN ELASTIC SOLID SUBJECTED TO GENERAL LOADING

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## FOREWORD

This report was prepared by the Division of Engineering, Brown University, Providence, Rhode Island, under USAF Contract No. F33615-71-C-1308. The contract was initiated under Project No. 7353, "Characterization of Solid Phase and Interphase Phenomena in Crystalline Substances," Task No. 735303, "Surface Effects and Mechanical Response." Funds for this project were supplied to the Air Force Materials Laboratory by the Office of Aerospace Research. The work was administered by the Metals and Ceramics Division, Air Force Materials Laboratory, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, with Dr. T. Nicholas, AFML/LLD, as Project Scientist.

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The author would like to gratefully acknowledge his colleague, Dr. J. R. Rice, for several helpful discussions.

This technical report has been reviewed and is approved.

A handwritten signature in black ink, appearing to read 'W. J. Trapp', with a stylized, cursive script.

W. J. TRAPP  
Chief, Strength and Dynamics Branch  
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# ABSTRACT

The stress intensity factor of a half-plane crack extending non-uniformly in an isotropic elastic solid subjected to general loading is determined. The loading is applied in such a way that a state of plane strain exists, and that crack extension takes place in mode I. The crack tip is initially at rest, and then moves in an arbitrary way in the plane of the crack. In the process of obtaining the stress intensity factor, the complete elastic field is determined for a crack which starts from some initial position, extends at a constant rate for some time, and then suddenly stops. Once the stress intensity factor is known for arbitrary motion of the crack tip, the Griffith fracture criterion is applied to obtain an equation of motion for the crack tip which is consistent with the assumptions of this criterion. Numerical results are included for the stress intensity factor and for the velocity-dependent term in the equation of motion.

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## 1. INTRODUCTION

Generally, the role of a fracture criterion in fracture mechanics is to select the actual motion of a crack tip for given environmental conditions from the set of all admissible motions. The criterion is a condition on some part of the field of the moving crack. In order to apply the criterion, this part of the field must be a known functional on the range of all admissible motions. The particular problem considered here is that of an elastic body containing a half-plane crack subjected to a general distribution of time-independent body forces and/or remotely applied tractions. The only condition on the applied loads is that the static elasticity solution exist for any amount of extension of the crack. The loading is taken to be uniform in the direction parallel to the edge of the crack, so that the state of deformation is one of plane strain. The loading is also assumed to be applied antisymmetrically with respect to the plane of the crack, so that extension will occur in mode I. The crack tip is at rest up until a certain instant, at which time the tip may begin to move. A means of determining the subsequent motion is sought.

The simplest fracture criterion available for studying the motion of a crack in an elastic body is that proposed by Griffith (see Rice, [1] for example), which states that all of the energy absorbed by the moving crack tip is subsequently stored in the form of surface energy of the newly created faces of the crack. If  $G$  is the energy release rate (Atkinson and Eshelby, [2]) which depends on the applied loads and the crack tip motion, and  $\gamma$  is the surface energy per unit area for the material at hand, then the actual motion of the crack tip satisfies

$$G = 2\gamma \quad . \quad (1.1)$$

It is the purpose here to determine the way in which  $G$  depends on applied loads and tip motion for the mode I problem outlined above. Once this dependence is known, then (1.1) may be viewed as an equation for determining the actual motion.

The analysis outlined here is essentially a continuation of recent work concerned with the constant velocity extension of a crack in a loaded elastic solid (Freund, [3]). For the most part, the notation established in that paper will be used here. In this earlier work, the crack is assumed to be at rest up until a certain instant, at which time it begins to extend at a constant rate defined by the velocity of the tip  $v$ . Before the crack extends, there is some time-independent normal stress distributed along the prospective fracture plane. By the process of crack extension is meant the total relaxation of this stress distribution.

The constant velocity problem is solved in two steps. First, it is assumed that there are no body forces or surface tractions initially acting on the body, and that the tip is moving with speed  $v$ . At a certain instant, concentrated normal forces of unit magnitude appear through the tip of the crack, one on each of the newly created faces. The tip then moves on along the fracture plane, but the forces continue to act at the same material point. A dynamic stress field, called the fundamental solution, is then generated. The second step is to use the fundamental solution in conjunction with linear superposition to construct a solution

for general loading. This is done by assuming that a traction distribution appears through the moving crack tip on each face which exactly negates the normal stress due to the applied loads. The reader is referred to the paper for the details.

The results of the constant velocity problem are now employed to determine the stress intensity factor and the energy release rate as functionals on the crack tip motion for the case of a non-uniform rate of extension. Unfortunately, no direct method is available for solving this problem. With some motivation provided by the results reported by Eshelby [4,5] for the extension of a crack in mode III at a nonuniform rate, however, it is possible to construct an inverse method to obtain the desired expressions for the stress intensity factor and energy release rate.

In Section 2 a solution is constructed for the particular problem of a crack which suddenly starts to extend at a constant rate, moves for some distance, and then suddenly stops. Certain details of this solution are studied in Section 3 which makes it possible to determine the stress intensity factor and energy release rate for the case of nonuniform motion.

## 2. Suddenly Stopping the Crack

Consider an unbounded elastic body containing a half-plane crack. The material is characterized by the dilatational and shear wave slownesses (inverse wave speeds)  $a$  and  $b$ . The slowness of Rayleigh waves is denoted by  $c$ . A Cartesian coordinate system is introduced in the body in such a way that  $z = 0$  is the plane of the crack, and the  $y$ -axis is parallel to the edge of the crack. The location of the tip is then specified

by its  $x$ -coordinate. Suppose that the tip is at  $x = 0$  at  $t = 0$ , and that no loads are applied to the body. At the instant  $t = 0$  concentrated forces of unit magnitude appear at the tip of the crack, one acting normally outward on each face at  $x = 0$ . For  $t > 0$  the crack tip moves in the positive  $x$ -direction with constant speed  $v = l/d$ , so that the  $x$ -coordinate of the tip is  $vt$ . As the forces do work a dynamic stress field is generated which is called the fundamental solution. This solution was obtained in the earlier paper, and will be used extensively here. It might be noted here that for the fundamental solution the faces of the crack penetrate each other. While this might be disturbing from a physical point of view, it in no way affects the analysis. The final results derived by means of the fundamental solution, however, are valid only for those physical situations in which the opposite faces of the crack do indeed move away from each other.

Suppose that the tip extends a distance  $l$  at the constant rate, and then suddenly stops, that is, the speed of the tip changes discontinuously from  $v$  to zero. The fundamental solution still solves the problem for  $vt < l$ , but some modification must be made for  $vt > l$ . For  $vt < l$  the governing equations and boundary conditions are given in the earlier work. For  $vt > l$ , the same governing equations must be satisfied, but their solution must now satisfy the boundary conditions

$$\sigma_{zz}(x, 0, t) = \delta(x), \quad x < l \quad (2.1)$$

$$\sigma_{xz}(x, 0, t) = 0, \quad -\infty < x < \infty \quad (2.2)$$

$$w(x, 0, t) = 0 \quad , \quad x > l \quad (2.3)$$

where  $\sigma_{ij}$  is the stress matrix,  $w$  is the normal component of displacement, and  $\delta$  is the Dirac delta function. In view of the symmetry with respect to the plane  $z = 0$ , consideration may be limited to the solution of the problem in the half-plane  $z \geq 0$ . The solution to this problem will be derived in an indirect way. Once the solution is obtained, however, it can be verified that it does in fact satisfy all of the necessary conditions.

Suppose for the time being that the tip does not really stop at  $x = l$ , but that it continues to move at constant speed  $v$ . Furthermore, suppose that for  $vt > l$  a time-independent distributed traction  $q(x)$  appears through the moving tip and acts on  $z = 0$ ,  $l \leq x \leq vt$ . The answer to the following question is now sought: What must  $q(x)$  be so that the stress intensity factor at the tip  $x = vt$  is zero for  $vt > l$ ? The reason for considering this particular question is that if there is to be any hope of treating the case of a nonuniformly moving tip by superposition of solutions of the type derived in the earlier paper, then a suddenly stopping crack must radiate out some sort of time-independent normal stress distribution in front of itself. This important point should become more clear as the discussion proceeds.

The answer to the question is obtained by making use of the expression (3.5) of the earlier paper for the stress intensity factor

$$K_v = \left(\frac{2}{\pi}\right)^{1/2} k(v) \int_0^{vt} \frac{1}{(vt-x)^{1/2}} p(x) dx \quad , \quad (2.4)$$

where a simple change of variable has been made in the integral. In (2.4),  $k(v)$  is a known function of speed  $v$  and  $p(x)$  is any time-independent normal traction distribution which appears through the moving tip on  $0 \leq x \leq vt$ . To determine the  $q(x)$  which makes  $K_v = 0$ ,  $p(x)$  is replaced in (2.4) by  $\delta(x) - q(x)$  and the right side of (2.4) is set equal to zero, with the result that

$$\int_{\ell}^{vt} \frac{1}{(vt-x)^{1/2}} q(x) dx = \frac{1}{(vt)^{1/2}}, \quad vt > \ell. \quad (2.5)$$

After a change of variable  $x' = x - \ell$  in the integral of (2.5), this equation takes the standard form for an Abel integral equation. The unique solution, given by Carrier, Krook and Pearson [7], p. 357, is

$$q(x) = \ell^{1/2} / \pi x(x - \ell)^{1/2}, \quad x > \ell. \quad (2.6)$$

The distribution (2.6) is immediately recognized as the "static" normal stress distribution on  $z = 0$ ,  $x > \ell$  in a body containing a half-plane crack occupying  $z = 0$ ,  $x < \ell$ , the faces of the crack being subjected to concentrated normal forces of unit magnitude (per unit distance in the  $y$ -direction) at  $x = 0$ ,  $z = 0^+$  and  $z = 0^-$ . The stress intensity factor for this distribution is  $(2/\pi\ell)^{1/2}$  at  $x = \ell$ .

It has therefore been established that if the concentrated unit loads appear through the moving crack tip at  $x = 0$  and the distribution  $-q(x)$  appears through the moving tip on  $x = 0^+$  for  $x > \ell$  (the opposite traction appears on  $z = 0^-$ ), then the stress intensity factor will be

zero at the tip  $x = vt$  for  $vt > l$ . This by no means implies that the crack has actually stopped at  $x = l$ , however, because it is not yet clear whether or not the boundary condition (2.3) is satisfied. It therefore remains to check to see if  $w(x, 0, t) = 0$  for  $x > l$ .

It turns out to be more convenient to consider the partial derivative of normal displacement with respect to  $x$  than to consider the displacement itself. By means of the superposition scheme discussed in the previous paper, the displacement gradient along  $z = 0$ , say  $\partial w_s / \partial x$ , may be written

$$\frac{\partial w_s}{\partial x}(x, 0, t) = \frac{\partial w}{\partial x}(x, 0, t) - \int_l^{vt} \frac{\partial w}{\partial x}(x-x_0, 0, t-x_0/v) q(x_0) dx_0, \quad (2.7)$$

where  $\partial w / \partial x$  is the displacement gradient of the fundamental solution. The substitution of  $\delta(x) - q(x)$  for the traction appearing through the tip has already been made. The expression (2.7) is now examined for  $l < x < vt$ . For convenience, distance measured positively behind the moving crack tip is denoted by  $\zeta = vt - x$ .

The displacement gradient of the fundamental solution is given by

$$\frac{\partial w}{\partial x}(x, 0, t) = \frac{t}{\pi \zeta^2} \operatorname{Im}[J_-(\frac{t}{\zeta})] H(t - a_1 \zeta) \quad (2.8)$$

for  $\zeta > 0$ . In (2.8),  $H$  is the unit step function and  $a_1 = a/(1 + a/d)$ . The function  $J_-(\lambda)$  is given by

$$J_{-}(\lambda) = \frac{C(d)(a_1 - \lambda)^{1/2}}{S_{-}(\lambda)(\lambda - c_1)(\lambda - d)} \quad (2.9)$$

where  $C$  is a constant depending on  $d$ , and  $c_1 = c/(1 + c/d)$ . The function  $S_{-}(\lambda)$  appearing in (2.9) has branch points at  $\lambda = a_1$  and  $\lambda = b_1 = b/(1 + b/d)$ , and is single-valued in the  $\lambda$ -plane cut between these points. The function  $S_{-}(\lambda)$  is analytic at all points except at the two branch points. Furthermore,  $S_{-}(\lambda) = O(1)$  as  $|\lambda| \rightarrow \infty$  so that  $J_{-}(\lambda) = O(\lambda^{-3/2})$  as  $|\lambda| \rightarrow \infty$ . The function  $J_{-}(\lambda)$  also has simple poles at  $\lambda = c_1, d$ . The relative magnitudes of the constants are such that  $a_1 < b_1 < c_1 < d$ .

The expression (2.8) is now substituted into the integral in (2.7). Making the change of variable in the integral  $x_0 = vt - \lambda v \zeta$ , the result is

$$\frac{\ell^{1/2}}{\pi^2 v^{1/2} \zeta^{3/2}} \int_{a_1}^{a^*} \text{Im} \left[ \frac{\lambda J_{-}(\lambda)}{(a^* - \lambda)^{1/2}} \right] \frac{1}{(\lambda - t/\zeta)} d\lambda \quad (2.10)$$

where  $a^* = (t - \ell d)/\zeta$ . For  $\ell < x < vt$ ,  $d < a^* < t/\zeta$ . Since  $J_{-}(\lambda)$  has poles on the path of integration, the integral in (2.10) must be interpreted in the principal value sense. Note that, since  $\lambda$  and  $(a^* - \lambda)^{1/2}$  are real for  $a_1 < \lambda < a^*$ , these factors are taken inside of the argument of the operator  $\text{Im}()$ .

The integral in (2.10) is a real integral. However, by viewing it as a line integral in the complex  $\lambda$ -plane, evaluation becomes a simple matter. For this purpose, the branch of  $(a^* - \lambda)^{1/2}$  which has a non-

negative real part everywhere in the  $\lambda$ -plane is chosen. The function  $\lambda J_{-}(\lambda)/(a^{*} - \lambda)^{1/2}$  is single-valued in the plane cut along  $\text{Im}(\lambda) = 0$ ,  $a_1 \leq \text{Re}(\lambda) \leq a^{*}$ . The integral (2.10) may then be viewed as a complex line integral along the upper side of this cut, as shown in Figure 1. Now, in view of the fact that  $\bar{\lambda} J_{-}(\bar{\lambda})/(a^{*} - \bar{\lambda})^{1/2} = \overline{\lambda J_{-}(\lambda)/(a^{*} - \lambda)^{1/2}}$ , where the bar denotes complex conjugate, the path of integration may be changed to a closed contour embracing the branch cut, as shown. (This appears to be a standard way of evaluating certain types of integrals in elasticity; see, for example, Green and Zerna [6], p. 274). The integral becomes

$$\frac{\ell^{1/2}}{\pi v^{1/2} \zeta^{3/2}} \frac{1}{2\pi i} \int_{\Gamma} \frac{\lambda J_{-}(\lambda)}{(a^{*} - \lambda)^{1/2}} \frac{d\lambda}{(\lambda - t/\zeta)} . \quad (2.11)$$

The only singularity of the integrand exterior to  $\Gamma$  is the simple pole at  $\lambda = t/\zeta$ . The integral (2.11) can be evaluated by Cauchy's integral theorem. The value of the integral taken along  $\Gamma$  is equal to the value taken along a circle of infinitely large radius plus the residue at  $\lambda = t/\zeta$ . Since the integrand is  $O(\lambda^{-2})$  as  $|\lambda| \rightarrow \infty$  the former contribution is zero. The value of the integral (2.11) is therefore

$$\frac{t}{\pi \zeta^2} \text{Im}[J_{-}(\frac{t}{\zeta})] . \quad (2.12)$$

If the result (2.12) is compared with (2.8), it is seen that the displacement gradient on  $z = 0^{+}$  due to the appearance of  $-q(x)$  through

the moving tip exactly cancels the displacement gradient of the fundamental solution for  $x$  in the range  $\ell < x < vt$ . Then  $\partial w_s(x, 0, t)/\partial x = 0$  in this interval, and  $w_s(x, 0, t)$  itself is a constant. Because  $w_s(vt, 0, t)$  is zero this constant is zero, and the boundary condition (2.3) is satisfied. The fundamental solution of the previous paper has therefore been extended to include the case of a crack which extends a distance  $\ell$  and then suddenly stops. If  $\phi_s$  is the dilatational displacement potential for this case, and  $\phi$  is the same potential for the fundamental solution, then

$$\phi_s(x, z, t) = \phi(x, z, t) - \int_{\ell}^{vt} \phi(x-x_0, z, t-x_0/v) q(x_0) dx_0. \quad (2.13)$$

Any other field variable for the suddenly stopping crack may be written in a similar form. The total field satisfies the appropriate equations of motion, conditions of rest for  $t = 0$ , the boundary conditions (2.1) - (2.3) of the earlier paper for  $vt < \ell$ , and the conditions (2.1) - (2.3) above for  $vt > \ell$ .

The foregoing analysis applies only in the case of concentrated loads appearing through the moving crack tip. Suppose now that the crack extends from  $x = 0$  to  $x = vt$  by negating an arbitrary normal stress distribution, say  $-p(x)$ ; that is, suppose a traction distribution  $p(x)$  appears through the moving tip. The function  $-p(x)$  defines the normal stress distribution on  $z = 0$ ,  $x > 0$  due to general applied loads on the body containing a crack occupying  $z = 0$ ,  $x < 0$ . As the moving tip passes  $x = \ell$ , an additional time-independent traction distribution, say  $-q^*(x)$ , begins

to appear through the tip, acting on  $\ell < x < vt$ . The answer to the following question is sought: For any  $p(x)$ , what must  $q^*(x)$  be so that the stress intensity factor is zero at  $x = vt$ ? The answer may be obtained in a variety of ways, the simplest of which appears to be linear superposition of the result in (2.6), which yields

$$q^*(x) = p(x) - \frac{1}{\pi(x-\ell)^{1/2}} \int_0^{\ell} p(x) (\ell-s)^{1/2} \frac{ds}{s-x}, \quad x > \ell. \quad (2.14)$$

The function  $-q^*(x)$  may be recognized as the normal stress distribution on  $z = 0$ ,  $x > \ell$  due to the same general loading applied to a body containing a crack occupying  $z = 0$ ,  $x < \ell$ .

As before, it can be shown that if the traction appearing through the moving crack tip is  $p(x) - q^*(x)$  then the normal displacement vanishes on  $\ell < x < vt$ , which implies that the crack tip actually stopped at  $x = \ell$ . Using the superposition scheme discussed in the previous paper, the complete solution for the sudden starting and stopping of a half-plane crack in a body subjected to general loading may be obtained. For example, if the general loading is defined by  $p(x)$ ,  $\ell$  is the total extension,  $v$  is tip speed,  $t = 0$  is starting time and  $\phi(x, z, t)$  represents the fundamental solution, then the dilatational displacement potential  $\phi_s^*$  is given by

$$\begin{aligned} \phi_s^*(x, z, t) = & \int_0^{\ell} p(x_0) \{ \phi(x-x_0, z, t-x_0/v) \\ & - \frac{1}{\pi} \int_{\ell}^{vt} \phi(x-\eta, x, t-\eta/v) \frac{(\ell-x_0)^{1/2} d\eta}{(\eta-\ell)^{1/2}(\eta-x_0)} \} dx_0. \end{aligned} \quad (2.15)$$

Similar expressions may be written for all field variables. This completes the solution for the sudden stopping of a crack.

As is usually the case in problems of this sort, the stress intensity factor is the quantity of primary interest. For the case of general loading, this quantity is given by (2.4) for  $vt < l$ , the stress being singular at  $x = vt$ . For  $vt > l$ , the stress was required to be non-singular at  $x = vt$ . This condition lead to a requirement that an additional traction distribution appear through the tip on  $l < x < vt$ , which turned out to be singular as  $x \rightarrow l^+$ . It was subsequently shown that this actually implied that the crack tip stopped at  $x = l$ , so that the stress singularity there actually defines the stress intensity factor for the suddenly stopped crack. This quantity, which will be called  $K_s(l)$ , is defined by

$$K_s(l) = \lim_{x \rightarrow 0^+} (2\pi)^{1/2} (x-l)^{1/2} q^*(x) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^l \frac{p(s)}{(l-s)^{1/2}} ds. \quad (2.16)$$

This is exactly the "static" stress intensity factor for the given applied loads when the crack tip is at  $x = l$ . Thus the interesting result is obtained that before the tip begins to move the stress intensity factor is  $K_0 \equiv K_s(0)$ . When the velocity changes from zero to  $v$ , its value changes from  $K_s(0)$  to  $K_v$  given in (2.4). And then, when the tip velocity changes from  $v$  to zero, the stress intensity factor changes discontinuously from  $K_v$  to  $K_s(l)$  and stays at that value. This encouraging result suggests the calculations of the next section.

### 3. Radiation of the Static Field

If there is to be any hope of applying the foregoing analysis to the problem of a crack extending at a nonuniform rate, then a time-independent stress field must radiate outward, at least along  $z = 0$ ,  $x > l$ , from the tip when it suddenly stops. The reason for this will be clarified in the next section. The result of the previous section that the stress intensity factor takes on an appropriate static value when a crack stops is necessary but not sufficient to guarantee the desired result. With regard to the case of mode III crack extension, Eshelby [5] has shown that when the crack tip stops the appropriate static field does indeed radiate outward, and is fully established behind a circular wavefront traveling with the shear wave speed and centered at the stationary crack tip. It appears unlikely that this strong result can be carried over to the case of mode I extension. Fortunately, the somewhat weaker result discussed below suffices.

Consider now the expression for displacement gradient (2.7) for  $x$  in the interval  $l - t/c < x < l$ . Then  $c_1 < a^* < d$ . The steps leading from (2.7) to (2.11) may once again be followed. The difference here is that for  $c_1 < a^* < d$  the integrand has two singularities exterior to  $\Gamma$ , namely, simple poles at  $\lambda = t/\zeta$  and at  $\lambda = d$ . The value of the integral is then the sum of (2.12) and the residue of the pole at  $\lambda = d$ . Making use of the form of  $C(d)$  in (2.9) which was reported in the previous paper, the residue is

$$\frac{\partial w_s}{\partial x}(x, 0, t) = \frac{(1-\nu)l^{1/2}}{\pi\mu x(l-x)^{1/2}}, \quad (3.1)$$

where  $\nu$  is Poisson's ratio and  $\mu$  is the shear modulus. The only result needed to determine (3.1) which is not given explicitly in the previous paper is a Taylor series expansion of the modified Rayleigh wave function  $R(\lambda)$ , defined in (2.15) of that paper, about  $\lambda = d$ . A fairly long calculation shows that the dominant term of that expansion is

$$R(\lambda) \approx -2(b^2 - a^2)(\lambda - d)^2.$$

The expression (3.1) is recognized as the static displacement gradient of the face  $z = 0^+$  of a half-plane crack  $z = 0$ ,  $x < \ell$  due to a concentrated load at  $x = 0$ . Since  $w_s(\ell, 0, t) = 0$  the displacement itself is the appropriate static displacement. If (2.7) is considered for  $x < \ell - t/c$ , an additional time-dependent term appears due to the simple pole of  $J_-(\lambda)$  at  $\lambda = c_1$  and the displacement does not correspond to the static displacement field. It is thus concluded that, if the constant velocity crack suddenly stops, the static displacement appropriate for the specified load and new length radiates out on the crack faces behind a point traveling with the Rayleigh wave speed. Although the details have been worked out here only for the case of the concentrated loads at  $x = 0$ , the conclusion is clearly true for the appearance of any traction distribution  $p(x)$  through the tip. This can be seen by examining the term in brackets in the superposition integral (2.15).

Next, the normal stress ahead of the crack is considered for  $vt > \ell$ . Following the discussion leading to (2.7), an expression for this stress is

$$\sigma_{zz}^*(x, 0, t) = \sigma_{zz}(x, 0, t) - \int_l^{vt} \sigma_{zz}(x-x_0, 0, t-x_0/v) q(x_0) dx_0, \quad (3.2)$$

where  $\sigma_{zz}(x, 0, t)$  is the normal stress for the fundamental solution. By an analysis paralleling that presented in discussing the displacement gradients, it can be shown that

$$\sigma_{zz}^*(x, 0, t) = \frac{1}{\pi x} \left( \frac{l}{x-l} \right)^{1/2} \quad (3.3)$$

for  $vt < x < l + (t - ld)/a$ . The expression (3.3) is recognized as the static stress distribution on  $z = 0$ ,  $x > vt$  for a half-plane crack occupying  $z = 0$ ,  $x < l$  subjected to concentrated surface loads at  $x = 0$ . Again, by means of the familiar superposition argument, the result carries over to the case of general loading. In the previous section it was shown that the function (3.3) also describes the appropriate stress distribution for  $l < x < vt$ . The important conclusion is then that, if a half-plane crack in a generally loaded body extends at a constant rate and then suddenly stops, the static normal stress distribution appropriate for the given applied loads and the new crack length radiates out along  $z = 0$  from the stationary crack tip behind a point moving with the dilatational wave speed  $a^{-1}$ . This result makes it possible to construct the stress intensity factor for a crack extending nonuniformly.

#### 4. Discussion

Consider an elastic body containing a half-plane crack  $z = 0$ ,  $x < 0$

subject to general time-independent loading applied antisymmetrically with respect to  $z = 0$  and independently of  $y$ . The applied loading gives rise to a static normal stress distribution on  $x > 0$ . Suppose that if the tip is at  $x = \ell$  rather than  $x = 0$  for the same given applied loads, then the static normal stress distribution on  $x > \ell$  will be denoted by  $-p(x; \ell)$ . Initially, the distribution is  $-p(x; 0)$ . Suppose that at time  $t = 0$  the crack begins to extend, and for  $t > 0$  the  $x$ -coordinate of the tip is given by the continuous, non-decreasing function  $\ell(t)$ . The function is restricted by the condition that  $\dot{\ell}(t) \leq 1$  for all time, that is, the speed of the tip is always less than the Rayleigh wave speed.

The curve  $\ell(t)$  versus  $t$  is now approximated by a polygonal curve  $L(t)$  with vertices lying on the original curve as shown in Figure 2. Successive times  $t_0 = 0, t_1, t_2, \dots$  are marked off, and the vertices of the polygonal curve are located at  $(0, \ell_0), (t_1, \ell_1)$ , and so on, where  $\ell_k = \ell(t_k) = L(t_k)$ . According to this approximation the crack tip moves with constant speed  $v_0 = (\ell_1 - 0)/(t_1 - 0)$  during time  $0 \leq t < t_1$ , it moves with constant speed  $v_1 = (\ell_2 - \ell_1)/(t_2 - t_1)$  during the time  $t_1 \leq t < t_2$ , and so on.

Suppose for the moment that the motion of the tip is actually described by the polygonal curve. The analysis of the preceding sections makes it possible to write an expression for the stress intensity factor as a functional of  $L(t)$ . Initially, the crack extends by negating  $-p(x; 0)$  on the prospective fracture surface, that is, by having the normal traction distribution  $p(x; 0)$  appear through the tip. From (2.4) the stress

intensity factor for  $t < t_1$  is given by

$$K_{v_0} = \left(\frac{2}{\pi}\right)^{1/2} k(v_0) \int_0^{v_0 t} \frac{1}{(v_0 t - x)^{1/2}} p(x; 0) dx \quad (4.1)$$

$$= k(v_0) K_s [L(t)]$$

where  $K_s[L(t)]$  is the static stress intensity factor for the given applied loads and a crack occupying  $x < L(t)$ . Suppose that at  $t = t_1 - 0$  the tip suddenly stops. According to the above analysis, the stress intensity factor suddenly changes to  $K_s(\ell_1)$  and the normal stress distribution  $-p(x; \ell_1)$  is radiated out along  $z = 0$ ,  $x > \ell_1$  behind a point moving with the dilatational wave speed.

In order to extend further the crack must negate  $-p(x; \ell_1)$  by having the traction distribution  $p(x; \ell_1)$  appear through the moving tip. Because of the remarkable fact that this distribution is time-independent the original solution may be applied once again. That is, at time  $t = t_1 + 0$  the crack begins to move with constant speed  $v_1$ . Then for  $t_1 < t < t_2$  the stress intensity factor is given by

$$K_{v_1} = \left(\frac{2}{\pi}\right)^{1/2} k(v_1) \int_{\ell_1}^{v_1 t} \frac{1}{(v_1 t - x)^{1/2}} p(x; \ell_1) dx \quad (4.2)$$

Referring to (2.14) and recognizing that  $p(x)$  is  $p(x; 0)$  and  $q^*(x)$  is  $p(x; \ell)$ , it can be shown by multiplying by  $(vt-x)^{-1/2}$  and integrating that

$$\int_{l_1}^{v_1 t} \frac{p(x; l_1)}{(v_1 t - x)^{1/2}} dx = \int_0^{v_1 t} \frac{p(x; 0)}{(v_1 t - x)^{1/2}} dx . \quad (4.3)$$

Therefore, (4.2) may be written

$$K_{v_1} = k(v_1) K_S[L(t)] . \quad (4.4)$$

This procedure may be continued indefinitely, with the result that in any interval  $t_n \leq t < t_n + 1$  the stress intensity factor is given by

$$K_{v_n} = k(v_n) K_S[L(t)] . \quad (4.5)$$

It is clear that the polygonal curve  $L(t)$  approaches  $l(t)$  as

$(t_{n+1} - t_n) \rightarrow 0$  for all  $n$ . It is assumed that the stress intensity factor for the motion  $L(t)$ , which is given exactly by (4.5), approaches the stress intensity factor for  $l(t)$  in the limit. The conclusion is then reached that the instantaneous value of the stress intensity factor for any motion of the crack tip depends on the tip motion only through the instantaneous total extension  $l(t)$  and the instantaneous velocity  $\dot{l}(t)$ . The stress intensity factor is given as a functional on the motion  $l(t)$  by

$$K(l, \dot{l}) = k[\dot{l}(t)] K_S[l(t)] , \quad (4.6)$$

where  $K_s(l)$  is the static value for a given amount of extension defined in (2.16). The function  $k$  is defined in the previous paper and, for the sake of completeness, a curve of  $k$  versus  $\dot{l}c$  is shown here in Figure 3 for Poisson's ratio  $\nu = 0.25$ . The form of  $K$  for mode I extension is the same as that obtained by Eshelby [5] for mode III extension. The function  $k$  is quite different in the two cases, however.

The form of the energy release rate  $G$  adopted here is that proposed by Atkinson and Eshelby [2]. Letting  $\dot{u}_i$  be the particle velocity vector and  $\tilde{E}$  be the total energy density (kinetic plus potential),  $G$  takes the form

$$vG = \lim_{\Delta \rightarrow 0} \int_{\Delta} (\sigma_{ij} \dot{u}_i n_j + v \tilde{E} n_1) ds, \quad (4.7)$$

where  $v$  is the instantaneous tip velocity,  $\Delta$  is a small circle centered at the tip, and  $n_i$  is the outward normal to  $\Delta$ . The first term in (4.7) represents the rate of work of the material exterior to  $\Delta$  on the material inside  $\Delta$ . The second term appears because  $\Delta$  is not a material surface, but is instead moving through the material. This term represents the net energy influx transported by the material through  $\Delta$  as the little circle moves through the body. The near-tip stress and velocity components are known up to terms of order  $[(x-vt)^2 + z^2]^{1/2}$  once the stress intensity factor and the instantaneous tip velocity are known. The stress distribution with angle around the tip takes the form shown in Equation (147) of the article by Rice [1]. Assuming that terms of this and higher

orders contribute nothing to (4.7), it is found that

$$G(\ell, \dot{\ell}) = \frac{1-\nu^2}{E} [K(\ell, \dot{\ell})]^2 A(\dot{\ell}) \quad (4.8)$$

where  $K$  is given by (4.6),  $A$  is a function defined in the previous paper, and  $E$  is Young's modulus.

Applying the fracture criterion (1.1) it is found that the actual motion of the crack is the solution of the nonlinear ordinary differential equation

$$\frac{2\gamma E}{(1-\nu^2)} [K_s(\ell)]^{-2} = F(\dot{\ell}) \equiv k^2(\dot{\ell}) A(\dot{\ell}) \quad (4.9)$$

The function  $F$  defined in (4.9) is plotted against  $\dot{\ell}c$  in Figure 4. From this figure it can be seen that if  $K_s$  is a decreasing (increasing) function of  $\ell$ , then the velocity will also be a decreasing (increasing) function of extension. For example, if a half-plane crack extends under the action of concentrated forces on its faces, the velocity will decrease from some finite value to zero at some total amount of extension, which may be calculated. This is an example of stable crack growth. The situation is quite different if  $K_s$  is an increasing function of  $\ell$ . Consider, for example, the motion of one tip of a crack of finite length due to uniform tension at infinity, in which case  $K_s \sim \ell^{1/2}$ . If the magnitude of the loading is sufficiently large, the crack tip will continue to accelerate from some finite initial velocity, at least up until that time at which

wave reflections from the stationary tip alter the field near the moving tip. It is tacitly assumed in this discussion that the crack is initially held at rest in some way while the loads, which are sufficiently large to cause extension, are slowly applied. The crack is then allowed to extend.

Much of the discussion of the mode III case presented by Eshelby [4, 5] applied to the present problem, and it will not be repeated here. Also, the brief discussion concerning crack branching in Section 4 of the previous paper would be more appropriate here than in relation to the case of constant velocity extension.

It is tempting to view (4.6) as a more general result than it really is. The analysis assumes a half-plane crack in an unbounded body, which precludes the possibility of reflected waves influencing the tractions which must be negated by the extending crack and, in particular, the stress singularity near the tip. That is, the quantity  $K_s$  in (4.6) must indeed be a "static" stress intensity factor, uninfluenced by dynamic loading. In certain problems, a fairly convincing argument might be presented about the small effect of reflected waves because of geometric decay, but this cannot be proposed as a general result. Some knowledge of the error introduced by neglecting wave reflections would be very useful in trying to extend the results of the present analysis.

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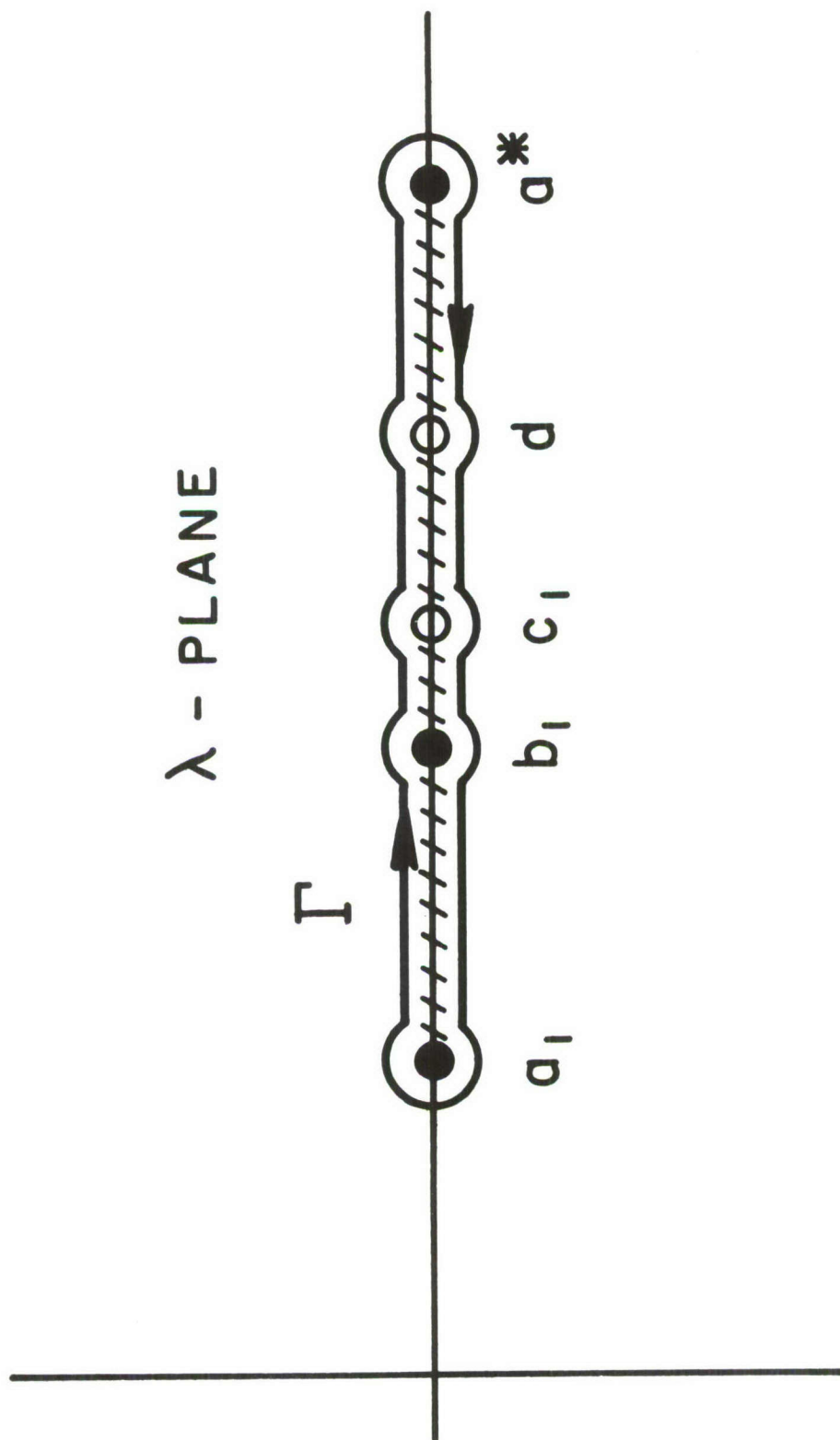


FIGURE 1

The complex  $\lambda$ -plane showing the integration path for the integral in (2.11).

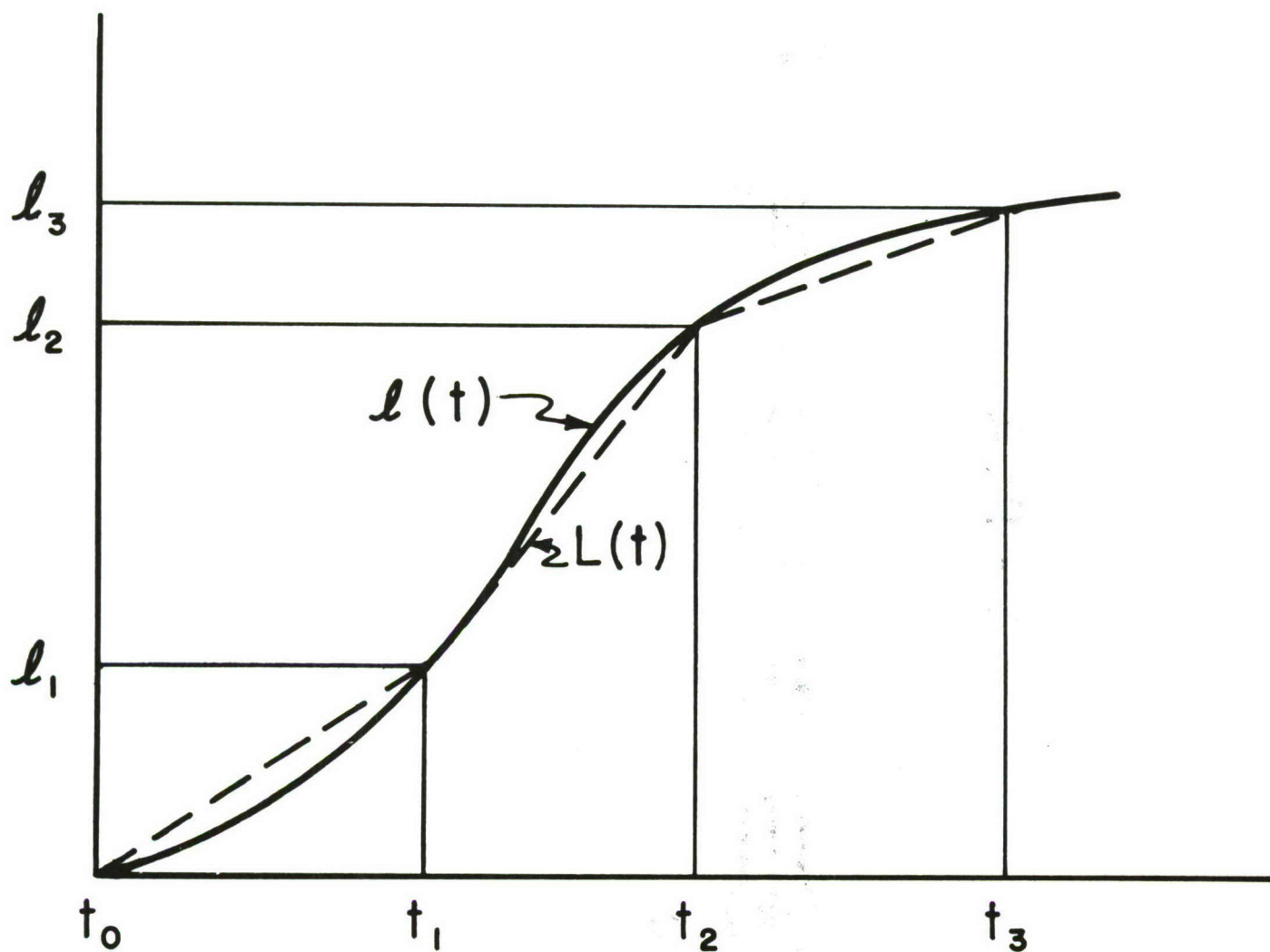
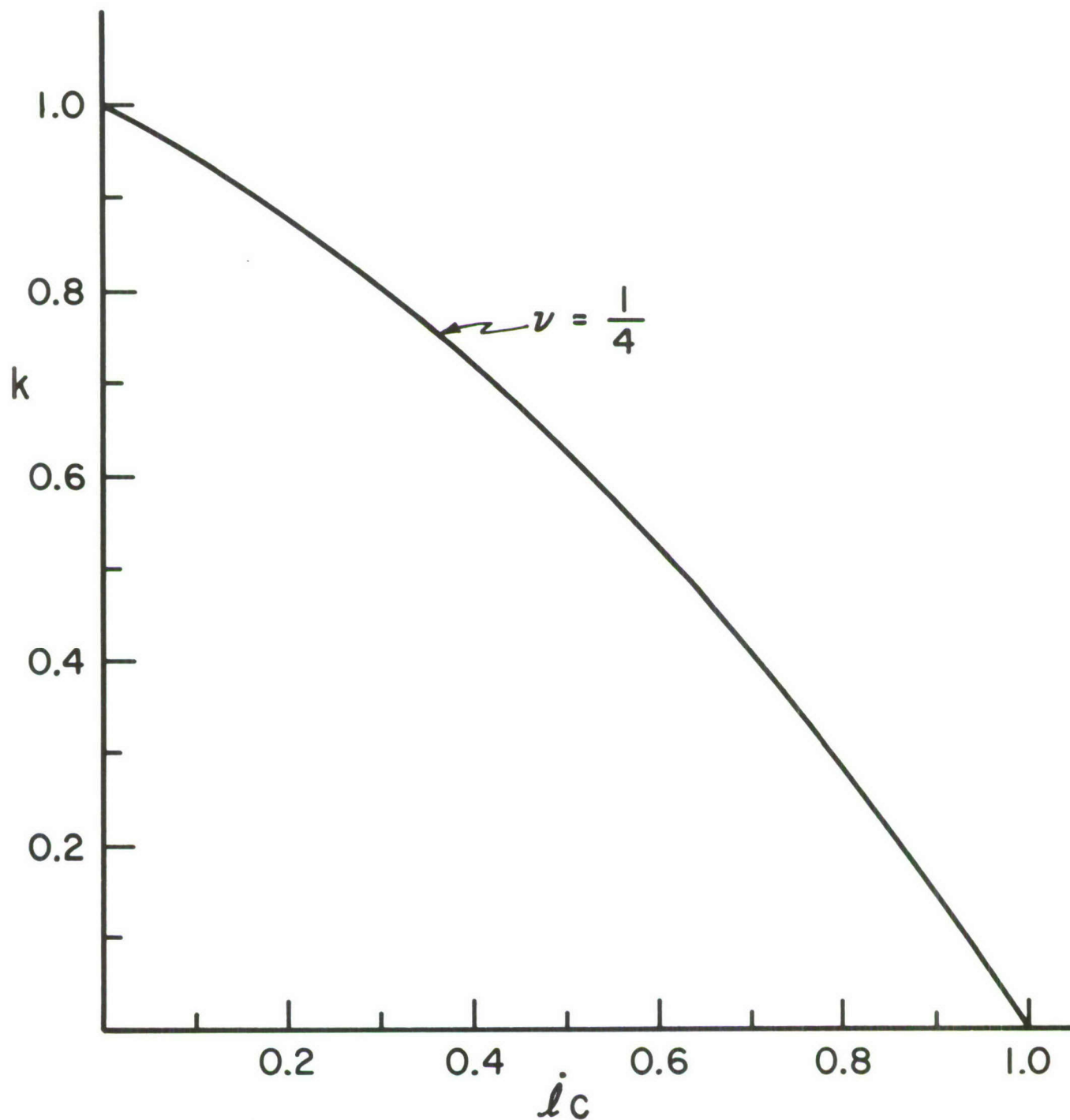


FIGURE 2

Polygonal approximation of a typical crack extension versus time curve.



**FIGURE 3**

A plot of  $k$ , defined in (4.6), versus dimensionless crack speed  $\dot{l}_c$  for  $\nu = 1/4$ .

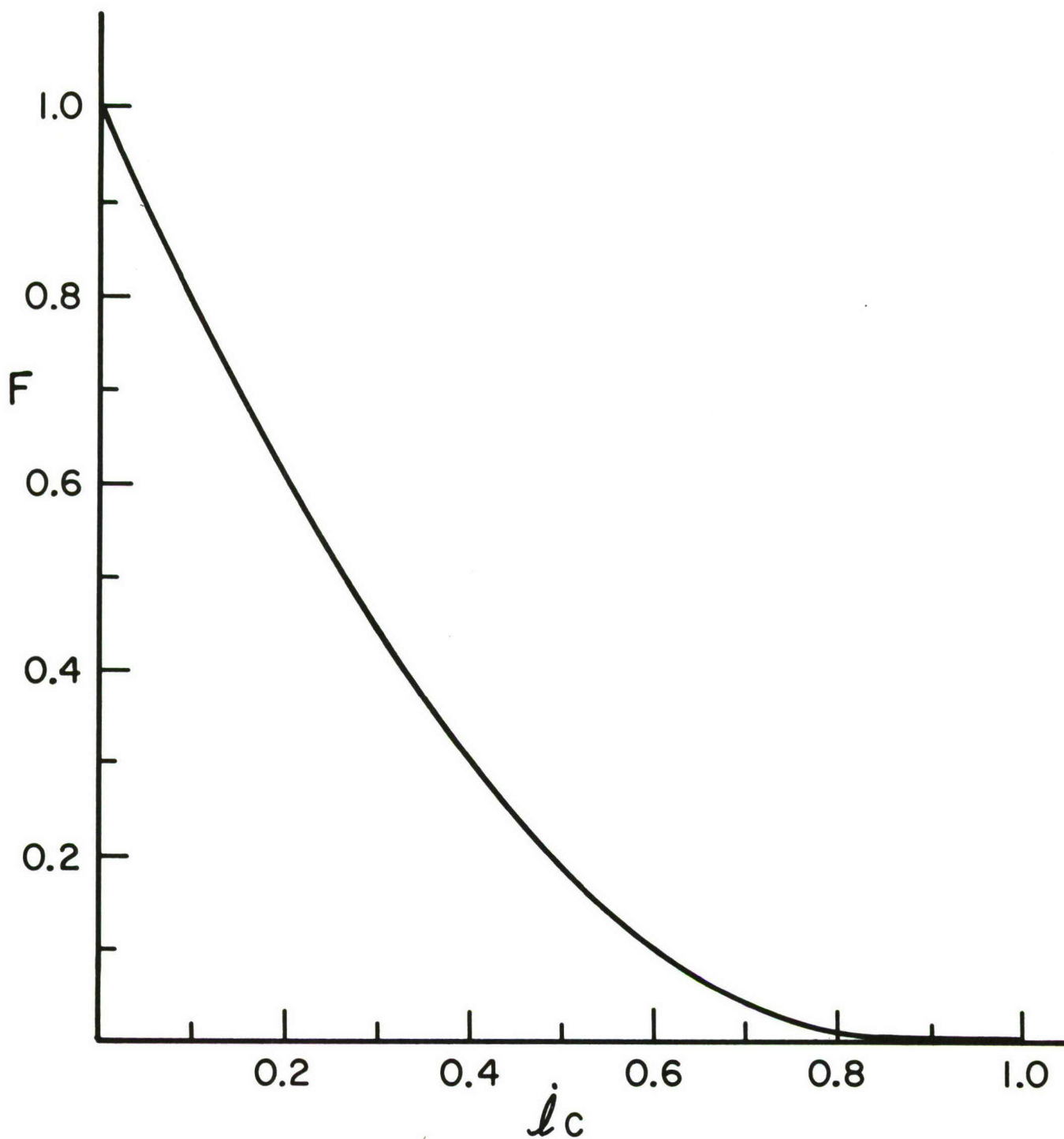


FIGURE 4

A plot of  $F$ , defined in (4.9), versus dimensionless crack speed  $\dot{l}_c$  for  $\nu = 1/4$ .

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13. ABSTRACT  The stress intensity factor of a half-plane crack extending non-uniformly in an isotropic elastic solid subjected to general loading is determined. The loading is applied in such a way that a state of plane strain exists, and that crack extension takes place in mode I. The crack tip is initially at rest, and then moves in an arbitrary way in the plane of the crack. In the process of obtaining the stress intensity factor, the complete elastic field is determined for a crack which starts from some initial position, extends at a constant rate for some time, and then suddenly stops. Once the stress intensity factor is known for arbitrary motion of the crack tip, the Griffith fracture criterion is applied to obtain an equation of motion for the crack tip which is consistent with the assumptions of this criterion. Numerical results are included for the stress intensity factor and for the velocity-dependent term in the equation motion.		

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